International Journal of Science Academic Research
Vol. 07, Issue 07, pp.1125-1128, July 2026
Available online at http://www.scienceijsar.com
ISSN: 2582-6425

Research Article

# PHASE PLANE METHODS TO FIND LIMIT CYCLE FOR NON - LINEAR DYNAMICAL SYSTEMS

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**Received** 13th May 2026; **Accepted** 18th June 2026; **Published online** 17th July 2026

**Abstract**

The phase plane method to findlimit cycle for non - linear dynamical systems is considered one of the important studies specifically in the domain of physics, engineering, economics, acquainted whether population of growth, administration and many of natural phenomenon so light is shed and this study aims to discuss a number of results concerning phase plane methods to find limit cycle in linear and non – linear dynamical systems, and to find fundamental review in phase plane methods, we followed the analytical induction mathematical methods , we found some the following results: The phase plane methods are graphical methods for non – linear systems , and they are the most general useful methods. and can find the limit cycle by graphical drawing.

**Keywords:** Linear dynamical systems, Graphical.

## INTRODUCTION

Where is non - linearsystems has an importance in physics, engineering, economics, acquainted whether population of growth, administration and many of natural phenomenon so light is shed an this study.h[1]Aset of two scalar different equations of the form:

$X^1(t) = F^1 (x(t) , y(t))$
$Y^1(t) = G^1(x(t) , y(t))$

as called a planar autonomous systems, such a planar of curve is called a trajectory of the system and its parameter interval is some maximal interval of existence $T_1<t<T_2$ where $T_1$ and $T_2$ might be infinite.

The graphic of a trajectory drawn as a parametric curve . In the x y graph is calledthe phase plane.

Linear system has closed orbit, but they won't be isolated (x(t))
Limit cycle is isolated closed trajectory (stable, anstable). Centre/Vertex Point:

**Stability of non-Linear Systems**

I. For Free System: A system is stable with zero input and arbitrary initial conditions ifthe resulting trajectory tends to the equilibrium state.
II. Force System: A system is stable if with bounded input, the system output is bounded.

There are types of stability condition in literature. Basically we concentrated on the following:

i. Stability, Asymptotic and Asymptotic in the large.
$x = (x)$ Asymptotic in the largeBecause $x(t_0) = 0$
$(t)$ remain near the origin for all $'t'$

ii. A system is said to be locally stable (stable in the small) if the resign $\delta(t)$ is small.

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 It is stable
 Every initial state $(t0)$ results in $X(t) \to 0$ as $t \to \infty$
 Asymptotically Stability in the large guarantees that every motion will approach theorigin.

**Phase Plane Methods**

Unforced linear spring mass damp system whose dynamics is described by

$$\mu\ \frac{d^2x}{dt^2} = 2\delta wn\frac{dx}{dt} + w_n^2 x\ = 0$$

$\delta$ is the damping factor and $\omega n$ =undamped natural frequency. Let the state of the system is described by two variables, displacement (x) and the velocity$\left(\frac{dx}{dt}\right)$in the state variable notation $x_1 = x$, $x_2 = \frac{dx}{dt'}$ ,

$\Rightarrow statevariables = phasevariables$

$\frac{dx_1}{dt} = x_2$

$$\frac{dx_1}{dt}\ = w_n^2 x_1 - 2\delta w_n x_2$$

$$\frac{dx_1}{dt}\ = x_2$$

$$\frac{dx_2}{dt}\ = -\frac{f}{\mu}x_2 - \frac{k_1k_2}{\mu}\ - \frac{k_2{k_1}^3}{\mu}$$

$\mu$ (if the spring force is NL then $k_1x + k_2x^3$)

 The coordinate plane with axes that corresponds to the dependent variables $x_1 = x$ andits first derivative $x_2 = \ddot{x}$ is called phase-plane.
 The curve described by the state point $(x_1, x_2)$ in the phase plane with time runningparameter is called phase-trajectory.
 A plane trajectory can be easily constructed by graphical or analytical technique.
 Family of trajectories is called phase-portrait.

The phase plane method uses state model of the system which is obtained from the differentialequation governing the system dynamics.

- This method is powerful generally restricted to systems described by 2nd order

Differential equation.

**The phase plane and trajectories:** We are studying systems of differential equations of the following form. Here, x, y both denote functions of a variable t (so really they are shorthand for x(t), y(t)

$x' = f(x, y)$
$y' = g(x, y)$

One way to understand a system like this is by trying to determine what the trajectories of the solutions look like in the phase plane. Recall that the phase plane is a plane with coordinates x and y, and a trajectoryis obtained by plotting all the points (x(t), y(t)) of some solution to the system.

There are two critical facts about trajectories in the phase plane.

- Through any point in the phase plane, there is a unique solution trajectory.
- No two solution trajectories can intersect.

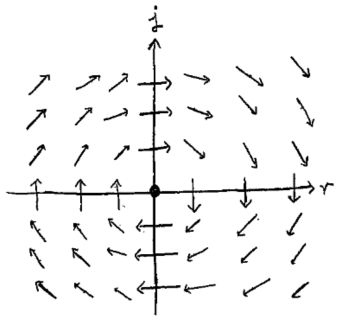
Also recall that there are equilibrium solutions, whose entire trajectories consist of a single point. Thesepoints can be identified as the values of x, y such that f (x, y) = 0 and g(x, y) = 0, so that the trajectorydoes not move at all.

For example, recall the toy example of Romeo and Juliet from last time.

$r'(t) = j(t)$
$j'(t) = -r(t)$

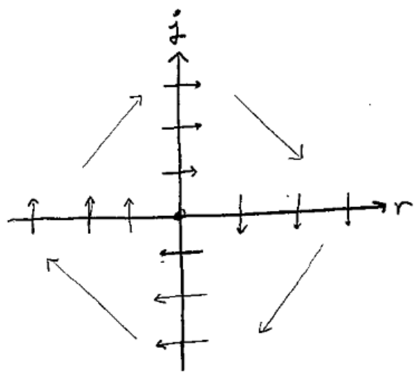
Then the trajectories in the phase plane consist of counter-clockwise circles, plus the single point at theorigin (which is an equilibrium solution).

If we do not yet know what the trajectories look like, one technique, which is entirely analogous to drawingslope fields1, is to observe that at any point (x, y), the values f (x, y), g(x, y) identify the direction in whichthe solution trajectory through (x, y) must be traveling. Indeed, the trajectory will have slope g(x, y)/f (x, y)(rise over run), unless f (x, y) = 0 and g(x, y) 6 = 0, in which case the trajectory will be perfectly vertical. For example, here is what happens if we draw arrows in various places in the phase plane for the Romeoand Juliet example, indicating which way trajectories must be traveling.




From this picture (especially if it were drawn more carefully), you could conjecture that the trajectorieswill move in counterclockwise circles.

Another, more readable way to encode some of this information is to simple draw the following picture.




Here, I have indicated the places where the trajectories must be perfectly horizontal (and whether theyare moving left or right) and the places where they must be perfectly vertical. This partitions the planeinto four regions. In each of these regions, I have drawn a single arrow to indicate which of the four general directions the trajectories must be traveling (up-right, up-left, down-left, or down-right). I can determinethis information just by looking at the signs of f (x, y) and g(x, y) in these regions.

While this second picture does convey less information (I know nothing about how steep the trajectoriesare), it turns out to still contain some useful information. More importantly, it is much easier to draw thesesorts of figures that the full direction field picture in more complicated situations. The following sections describe the general process for doing this .

**Limit Cycles**

System Stability has been defined in terms of distributed steady- state coming back to itsequilibrium position or at least staying within tolerable limits from it. Distributed NL system while staying within tolerable limits may exhibit a special behavior of closed trajectory/limit cycle.

The limit cycles describe the oscillations of non-linear system. The existence of a limit cycle corresponds to an oscillation of fixed amplitude and period.

**Example (5.1):**

Let us consider the well- known Vander Pol's differential equation

$$\frac{d^2x}{dy^2} - \mu(1 - x^2)\frac{dx}{dy} + x = 0$$

It describes the many non-linearities by comparing with the following linear differentialequation:

$$\frac{d^2x}{dy^2} + 2\delta\frac{dx}{dy} + x = 0$$

$2\delta = -\mu(1 - x^2) \rightarrow \delta = \left(\frac{\mu}{2}\right)(1 - x^2)$ {Damping Factor}

a. $|x| \gg 1$, then damping factor has large positive means system behaves overdampedsystem with decrease of the amplitude of $x1$. In this process the damping factor alsodecreases and system state finally reaches the limit cycle.

b. $|x| \ll 1$, then damping factor is negative, hence the amplitude of x increases till thesystem state again enters the limit cycle.

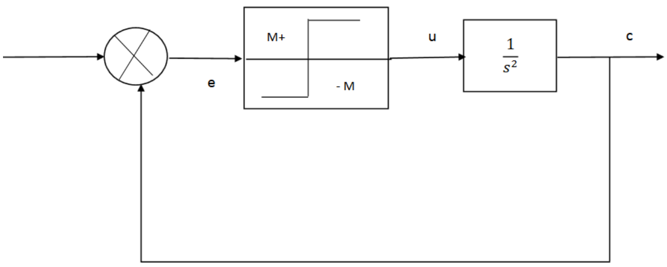
 On the other hand, if the paths in the neighborhood of a limit cycle diverse awayfrom it, it indicates that the limit cycle is unstable.

**Example (5.2):**

Vander Pol Equation

$$\frac{d^2x}{dy^2} + \mu(1 - x^2)\frac{dx}{dy} + x = 0$$

 It is important to note that a limit cycle in general is an undesirable characteristics of acontrol system. It may be tolerated only if its amplitude is within specified limits.
 Limit cycles of fixed amplitude and period can be sustained over a finite range ofsystem parameter.




The differential equation describes the dynamics of the system which is given by

$c = u$
$r - c = e$
$-\dot{c} = \dot{e} \Longrightarrow -\ddot{c} = \ddot{e}$ (as $r$ constant)

Therefore $e = -u$
Choosing the state vector $[x_1 x_2] = [e e]^T$, then we have

$\dot{x_1} = x_2$
$\dot{x_2} = \ddot{e} = -u$

The output of on/off controller is
$u = \mu sig(e) = \mu sign(x_1)$

The state equation is described as
$\dot{x_1} = x_2$
$\dot{x_2} = -\mu sg(x_1)$

The slope of the equation is given b

$$\frac{dx_2}{dx_1} = \frac{\mu\, sign(x_1)}{x_2}$$

Separating the variables and integrating, we get

$$\int_{x_{2(0)}}^{x_2} x_2 dx_2 = -\mu \int_{x_1(0)}^{x_1} sign(x_1)dx$$

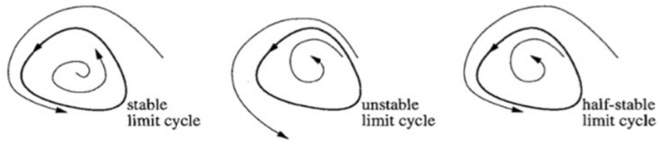
Carrying out integration and rearranging we get
$x_2^2 = 2\mu x_1(0) + x_2^2(0)$; $x_1 < 0$

Region ($x_1 = epositive, relayouput + \mu$)
$x_2^2 = 2\mu x_1 - 2\mu x_1(0) + x_2^2(0)$; $x < 0$

Region II ($x_1 = eneagtive, relayoutput-$ ) .

A limit cycle is an isolated closed trajectory: neighboring trajectories either spiral away from it or towards it.

If neighbouring trajectories tend towards the limit cycle, the latter is called stable or attracting, otherwise unstable, in exceptional cases it may be half-stable .

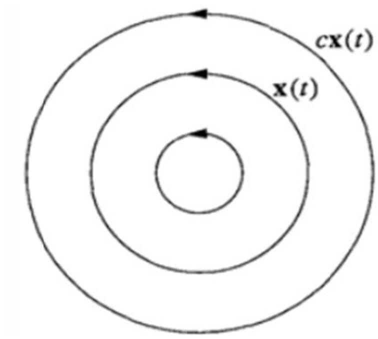



Stable limit cycles model systems/phenomena that exhibit self-sustained oscillations, like:

1. The beating of a heart
2. The periodic firing of a pacemaker neuron
3. Daily rhythms in human body temperature and hormone
4. secretion
5. Chemical reactions that oscillate spontaneously
6. Dangerous self-excited vibrations in bridges and airplane wings.

A slight perturbation makes the system return to the cycle. Limit cycles are typical features of nonlinear systems: in linear systems there are periodic orbits, but they are not isolated!

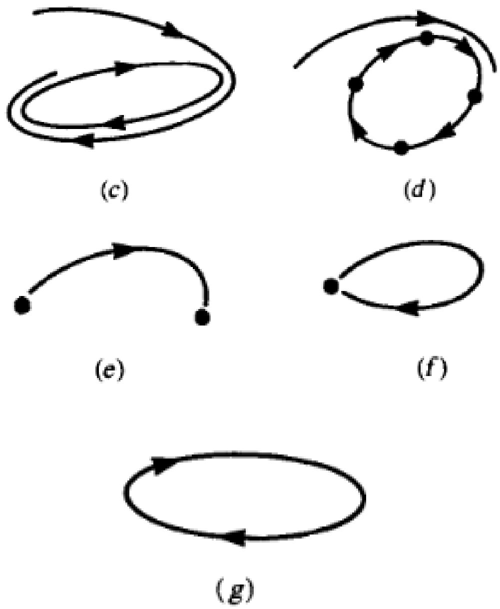
If x(t) is a periodic solution of a linear system, cx(t) is also a solution, for any value of c.




The amplitude of a linear oscillation is set by its initial condition: any perturbation will persist forever. Limit cycles are determined by the structure of the system itself.**6.Limit sets described in text:**

(a) steady-state point, (b) infinity, (c) closed-loop trajectory, (d) cycle graph, (e) heteroclinictrajectory, (f) monoclinic trajectory, and (g) limitcycle.

(a) (b)




## RESULTS

After we showed the phase plane method to find limit cycle for linear and non-linear dynamical systems we found the following results:

The phase plane method is graphical methods for linear and non-linear systems, and it is most general useful methods. We find the phase plane methods and limit cycle by graphical drawing.

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